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ADAPTION OF KOITER'S METHOD TO FINITE ELEMENT ANALYSIS OF SNAP-THROUGH BUCKLING BEHAVIOR†

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Abstract—Koiter's approach to the analysis of load–displacement behavior in the neighborhood of a bifurcation buckling point is adapted to permit the analysis of behavior in the neighborhood of the buckling point for a structure that exhibits snap-through buckling. The essential concept is the consideration of pre-buckling nonlinearities as generalized initial imperfections of a derived perfect structure called the modified structure. The asymptotic character of Koiter's approach is preserved in this modified structure method of analysis. The development is stated in a functional notation and is then discretized into a matrix procedure based upon finite element idealization. Numerical results are presented for the load–displacement behavior of a circular arch.

1. INTRODUCTION

(a) *Background*

PRESENT structural analysis practices, particularly within the aerospace industry, rely heavily upon the matrix methods of structural analysis based upon finite element idealizations. Highly organized computer programs of broad applicability are used extensively for the prediction of linear behavior in support of the structural design process.

The design of a major portion of the primary structure of flight vehicles involves consideration of structural stability and, with rare exceptions, structural stability phenomena are essentially nonlinear. That is, the load-displacement behavior is nonlinear prior to buckling, and buckling is usually of the snap-through rather than of the bifurcation type. Reference [1] presents a comprehensive review of the finite element methods that have been developed for the prediction of nonlinear behavior. Unfortunately, these methods require very extensive amounts of calculation to predict nonlinear behavior.

Practical considerations of structural stability usually proceed, necessarily, on the basis of a linearized (eigenvalue) analysis. The deficiency of this approach is that the error in the buckling load is often substantial (for example, see [2]) and no information is obtained about nonlinear behavior. A method for the analysis of stability of structures is needed which is more complete and more accurate than a linearized (eigenvalue) analysis and which is more efficient than a direct attack on the prediction of nonlinear behavior. The method presented in this paper is of such a nature.

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The foundation for this method of analysis was set down by Koiter in his fundamental work on the stability of equilibrium [3]. The principle of minimum potential energy is used to derive an asymptotic technique that gives an approximation for initial post-buckling behavior and imperfection sensitivity of structures. Koiter's technique, and parallel methods that have been developed independently by other researchers, have been used to investigate the behavior of a number of structures [4–12].

It is significant that Koiter's technique is applicable only to structures which exhibit a bifurcation point type of buckling. This mode of buckling is usually found in a structure which is symmetric both in geometry and in the applied load system. The applied load system is assumed proportional to a non-negative load parameter λ. For small values of λ, the displacements of the structures are continuous functions of λ and preserve the symmetry. Usually, as λ increases, there is a critical point where asymmetric displacements become possible. The load–displacement curve up to the critical point is called the fundamental equilibrium path. A bifurcation of the fundamental equilibrium path occurs at the critical point. The association between symmetry and bifurcation is reflected in the terminology, and a structure having a bifurcation point is often referred to as a "perfect" structure.

A prerequisite to the use of Koiter's method is the determination of the fundamental equilibrium path of the perfect structure U, as an explicit function of the applied load parameter λ. Unless this fundamental equilibrium path is very simple, the analytical application of Koiter's method becomes a formidable task. For this reason, application of Koiter's method has been made almost exclusively to structures with trivial or linear fundamental equilibrium paths.

Starting with knowledge of the fundamental path, the first part of Koiter's analysis is initiated by looking for points of possible instability on that path. Stability is equated with a strict minimum of the potential energy. This means that, if $U(\lambda)+u$ is any kinematically admissible state adjacent to $U(\lambda)$, then for stability, given any nontrivial u,

$$\Pi[U(\lambda)+u]-\Pi[U(\lambda)] \equiv P(u) > 0, \tag{1}$$

where Π denotes the total potential energy functional. The transition functional $P(u)$ is expanded in a Taylor series about $u = 0$. A critical point is one where the quadratic terms of $P(u)$ (the second variation of Π) become positive semidefinite. This point may be stable, neutral or unstable depending on the higher-order terms in the expansion of $P(u)$. In the case of critical points which are bifurcation points, Koiter showed how to use the Taylor expansion of $P(u)$ to predict post-buckling behavior.

The second part of Koiter's method is to consider a class of structures which differ slightly from the original. The difference or "imperfection" that determines this class usually destroys the symmetry of the structure, and the asymmetric structures do not have bifurcation points—they are "imperfect". Nevertheless, the results of the first part of the Koiter method are readily corrected to account for the imperfection effects. Snap-through buckling loads are obtained which are asymptotically exact for vanishingly small imperfections. Generally, predictions are satisfactory because imperfections in practical structures are sufficiently small.

The principal virtue of Koiter's method of analysis is that, after the fundamental path has been determined, the remaining nonlinear problem is transformed into a sequence of linear problems. Because interest is confined to the region of the buckling load, the nonlinear analysis is reduced to an effort equivalent to one or two linear analyses. This is only slightly

more effort than a linearized stability analysis and is far less effort than a direct nonlinear behavior analysis.

Improvement of the practices of finite element analysis on the basis of this efficient asymptotic method has small utility because the class of structures which undergo bifurcation has extremely few members. Many important problems of this class are amenable to analytical solutions, obviating the need for finite element solutions. This situation attaches particular significance to the adaption of the Koiter method to permit analysis of load–displacement behavior in the neighborhood of a snap-through buckling load.

(b) *The modified structure method*

Recognizing pre-buckling nonlinearity as an essential characteristic of snap-through buckling behavior, one is led to linearization of the fundamental load–displacement behavior as a means of accomplishing this transformation. Applicability of the Koiter method of analysis to problems with a nonlinear snap-through type of buckling is accomplished by transforming these problems into linear bifurcation buckling problems. A hypothetical structure, for which the linearized analysis yields the fundamental equilibrium path, is created by modifying the potential energy functional for the actual structure. This hypothetical or *modified* structure generally undergoes bifurcation buckling, and it will be referred to as the "perfect" structure (in Koiter's method, perfectness corresponds to having a bifurcation point). The difference between the potential energies of the actual imperfect structure and the hypothetical perfect structure is proportional to the terms that are responsible for the nonlinear behavior. Thus, the "imperfection" is in a sense proportional to the nonlinearity of the actual structure. Because of other conditions governing the design of actual structures using stiff materials, most structural behavior is only moderately nonlinear in the design range, and thus the "imperfections" to the modified structure are small.

This concept of a modified structure that is "perfect" has value only if the analysis permits recovery of the behavior of the actual structure. Indeed, this is the case. Account of the effect of pre-buckling nonlinearities proves to be possible within the theoretical framework set down by Koiter for the consideration of initial imperfections. Thus, the adaption to snap-through buckling problems is accomplished within the framework of Koiter's theory [3].

The statement of the modified structure method of analysis in Section 2 is followed in Section 3 by expression of the method in terms of finite element matrices. Previous applications of the theory of Koiter [3] have sought analytical results on the basis of continuum formulations. Such efforts are highly oriented to particular problems. The generality of the finite element approach adopted herein is in marked contrast to these previous developments.

The discretization and matrix abstraction of the finite element statement of the modified structure method of analysis renders computations independent of particular problems and particular types of structures. Verification presented in Section 4 for the particular case of a circular arch establishes the modified structure method both conceptually and procedurally for other types of structures as well. In principle, it remains only to furnish the appropriate types of finite element matrices.

The form of the finite element representation employed herein is drawn from [13]. It is significant that the finite element matrices already developed and employed in linear (eigenvalue) stability analyses and nonlinear behavior analyses are adequate for the

modified structure method of analysis. Thus, existing capabilities for linear stability analyses and nonlinear behavior analyses can be readily adapted to execute the modified structure analysis procedure.

The difficulty of obtaining solutions to partial differential equations has confined prior applications of Koiter's theory almost exclusively to the prediction of first-order approximations to post-buckling equilibrium paths. Even first-order continuum solutions become exceedingly difficult, however, when using the modified structure method. On the other hand, the discretized finite element implementation of the modified structure method of analysis is not difficult. Indeed, higher-order approximations are obtained with relative ease. Advantage is taken of this and results for such higher-order approximations are presented for the arch structure in Section 4.

Further developments of the present method of analysis is expected to extend its applicability to more highly nonlinear structures. This is shown in Fig. 1 wherein the

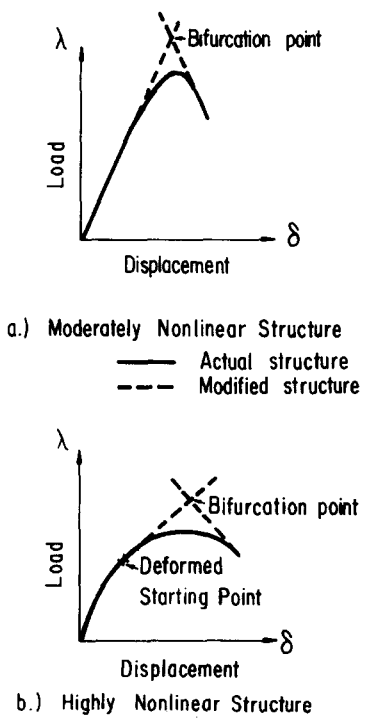


FIG. 1. Schematic illustration of modified structure method.

asymptotic analysis is illustrated as being initiated from a deformed state on a nonlinear fundamental equilibrium path. This approach is attractive because the asymptotic method tends to be most accurate in the region of the maximum load, and this is precisely the region where the direct prediction of behavior is the most difficult.

2. ANALYSIS METHOD IN FUNCTIONAL FORM

(a) *Basic notation*

Fidelity to the notation of [3] is maintained in order that results from that source may be used without repeating the underlying derivation. In this way, conciseness is achieved in describing the method without sacrificing completeness.

The statement of the total potential energy functional introduces the principal symbolic notation. We denote by U the set of displacement functions and displacement function derivatives which appear in the total potential energy functional $\Pi(U)$ for a given structure and given loads. The functional $\Pi(U)$ is expressed as a sum of functionals, each of which is homogeneous of degree i in the components of U, i.e.

$$\Pi(U) = \Pi_0(U)+\Pi_1(U)+\Pi_2(U)+ \ldots \tag{2}$$

where, for any scalar "a",

$$\Pi_i(aU) = a^i\Pi_i(U) \qquad i = 0, 1, 2, \ldots. \tag{3}$$

Extending this symbolic notation, the bilinear functional Π_{11} is defined by

$$\Pi_2(U+V) = \Pi_2(U)+\Pi_{11}(U, V)+\Pi_2(V). \tag{4}$$

For example $\Pi_2(U) \equiv cU^2$ yields $\Pi_{11}(U, V) = 2cUV$. This notation is usefully generalized to define various additional functionals, e.g.

$$\Pi_3(U+V) = \Pi_3(U)+\Pi_{21}(U, V)+\Pi_{12}(U, V)+\Pi_3(V), \tag{5}$$

and

$$\begin{aligned}\Pi_3(U+V+W) = {} & \Pi_3(U)+\Pi_{21}(U, V) \\ & +\Pi_{21}(U, W)+\Pi_{12}(U, V)+\Pi_{12}(U, W) \\ & +\Pi_{21}(V, W)+\Pi_{12}(V, W)+\Pi_3(V)+\Pi_3(W) \\ & +\Pi_{111}(U, V, W).\end{aligned} \tag{6}$$

Each functional is of homogeneous degree in the variables as indicated by the subscripts. Thus, $\Pi_{21}(U, V)$ is comprised of cubic terms which are of second degree in the components of U and of first degree in the components of V. For example, $\Pi_3(U) \equiv cU^3$ yields $\Pi_{21}(U, V) = 3cU^2V$ and $\Pi_{12}(U, V) = 3cUV^2$.

(b) *The perfect structure*

The equation of equilibrium can be derived by setting the first variation of the total potential energy of equation (2) equal to zero. Thus the variational equation is

$$0 = \Pi_1(\delta U)+\Pi_{11}(\delta U, U)+\Pi_{12}(\delta U, U)+\Pi_{13}(\delta U, U)+ \ldots. \tag{7}$$

Clearly, all functionals beyond the first two contribute to the equilibrium equation terms that are nonlinear in the components of U. The applied load is assumed to appear in the first term on the right of equation (7) as the product of a unit load system and the non-negative load intensity parameter λ.

Application of the method of [3] requires knowledge of the solution of the equilibrium equations, with boundary conditions, as an explicit function of λ for λ increasing from zero.

In general, such a solution is difficult to obtain; on the other hand, linearization of equation (7) gives the form

$$0 = \Pi_1(\delta U) + \Pi_{11}(\delta U, U) \tag{8}$$

that readily admits a solution of the form†

$$U(\lambda) = U_l \equiv \lambda U_0 \tag{9}$$

where U_0 is the solution of equation (8) for the unit load system.

Obviously, this solution is not generally an extremum of the potential energy functional $\Pi(U)$. Indeed, it is easy to construct a potential energy functional $\bar{\Pi}(U)$ for which the $U(\lambda)$ in equation (9) is the extremum; this is

$$\bar{\Pi}(U) = \Pi(U) - Q_1(U) \tag{10}$$

where

$$Q_1(U) = \Pi_{12}(U, \lambda U_0) + \Pi_{13}(U, \lambda U_0) + \Pi_{14}(U, \lambda U_0) + \ldots. \tag{11}$$

It is interesting to note that, because $Q_1(U)$ is linear in U, it can be interpreted as an external load energy, and the potential energy $\bar{\Pi}(U)$ could derive from a rather special external load system. Examination of equation (11) discloses that the form of $Q_1(U)$ is identical with that which arose in [3] in accounting for effects of initial imperfections. The only difference is that, in Koiter's work, $Q_1(U)$ is multiplied by a parameter of imperfection ε which can be made as small as desired. Here we must rely on an inherent smallness of $Q_1(U)$ itself. It follows that the pre-buckling nonlinearities can be accounted for by using the same technique that Koiter introduced for initial imperfection effects. With this observation, the conceptual generalization of Koiter's method to the analysis of load–displacement behavior in the neighborhood of a snap-through buckling load is accomplished. It remains to describe the procedural aspects.

Hereinafter, consistent with [3], $\bar{\Pi}(U)$ will be taken to define the "perfect" structure. The $U(\lambda)$ of equation (9) will be designated the "fundamental" solution, and the nonlinear terms which comprise the energy contribution $Q_1(U)$ will be treated as generalized initial imperfections.

(c) *The bifurcation point*

The analysis is carried forward by calculation of the bifurcation buckling load of the perfect structure. Toward this end, the potential energy of transition from the fundamental state to an adjacent state is formed by analogy with equation (1). Use of a Taylor series expansion yields this transition energy naturally as a sum of contributions of homogeneous order. From

$$P(u) = \bar{\Pi}(\lambda U_0 + u) - \bar{\Pi}(\lambda U_0) \tag{12}$$

we obtain, for each u sufficiently small,

$$P(u) = P_1(u) + P_2(u) + P_3(u) + \ldots \tag{13}$$

† It is tacitly assumed here that Π_1 is linear in λ, and that Π_2 is not dependent on λ. If this is not so, linearization entails the dropping of some parts of Π_1 and Π_{11}, and a corresponding change in Q_1. [See the example of Section 4(a).]

where the homogeneous contributions $P_j(u)$ may be expressed in terms of notation previously defined:

$$P_1(u) = 0 = \Pi_1(u) + \Pi_{11}(u, \lambda U_0), \tag{14}$$

$$P_2(u) = \Pi_2(u) + \Pi_{21}(u, \lambda U_0) + \Pi_{22}(u, \lambda U_0) + \ldots, \tag{15}$$

$$P_3(u) = \Pi_3(u) + \Pi_{31}(u, \lambda U_0) + \Pi_{32}(u, \lambda U_0) + \ldots, \tag{16}$$

$$P_4(u) = \Pi_4(u) + \Pi_{41}(u, \lambda U_0) + \Pi_{42}(u, \lambda U_0) + \ldots, \tag{17}$$

etc.

Upon increase of λ from zero, $P_2(u)$ is positive-definite in u for sufficiently small λ. At the bifurcation buckling load level λ_1, $P_2(u)$ vanishes for u which satisfies

$$P_{11}(\delta u, u) = 0. \tag{18}$$

Equation (18) is written equivalently as

$$0 = \Pi_{11}(\delta u, u) + \Pi_{111}(\delta u, u, \lambda U_0) + \Pi_{112}(\delta u, u, \lambda U_0) + \ldots. \tag{19}$$

Solutions of this nonlinear eigenvalue equation exist only for certain values of the load intensity parameter. Herein, it is assumed that the lowest eigenvalue $\lambda = \lambda_1$ is discrete and has the unique eigenfunction $u = u_1$.

(d) *Approximate behavior analysis*

The behavior of the actual imperfect structure is now approached by expansion of the potential energy about the bifurcation point of the perfect structure. Account is taken of applied load variations $(\lambda - \lambda_1)$ in this expansion, and the generalized initial imperfection terms are included. From equations (9), (10) and (13), and with use of the prime (') to denote differentiation with respect to the load, we obtain

$$\begin{aligned}
\Pi(U, \lambda) &= \bar{\Pi}(\lambda U_0, \lambda) + P(u, \lambda) + Q_1(\lambda u_0, \lambda) + Q_1(u, \lambda) \\
&= \bar{\Pi}(\lambda U_0, \lambda) + Q_1(u) + (\lambda - \lambda_1) Q_1'(u) + (\lambda - \lambda_1)^2 Q_1''(u) + \ldots \\
&\quad + P_2(u) + (\lambda - \lambda_1) P_2'(u) + (\lambda - \lambda_1)^2 P_2''(u) + \ldots \\
&\quad + P_3(u) + (\lambda - \lambda_1) P_3'(u) + (\lambda - \lambda_1)^2 P_3''(u) + \ldots \\
&\quad + P_4(u) + (\lambda - \lambda_1) P_4'(u) + (\lambda - \lambda_1)^2 P_4''(u) + \ldots,
\end{aligned}$$

etc. (20)

The Euler equations of this potential energy functional are sought in two stages by taking

$$u = a u_1 + \bar{u}. \tag{21}$$

First, "a" is held constant and $\bar{u}$ is determined. Then, the remaining single degree of freedom problem in "a" is solved to complete the behavior analysis.

Upon substitution of equation (21) into (20) to obtain the energy functional in terms of the chosen unknown field, $\bar{u}$, it becomes necessary to truncate the energy functional expansion in order to find a solution. Moreover, the nonlinearity of the Euler equations for $\bar{u}$ necessitates the approximation of this unknown field by a series expansion. This derivation

is presented in detail in [30, p. 85]. We record the resulting solution for $\bar{u}$ as

$$\bar{u} = \varphi_0 + \varphi_1' a(\lambda - \lambda_1) + \varphi_2 a^2 \tag{22}$$

in which terms cubic and higher [i.e. $a^i(\lambda-\lambda_1)^j$, for $i+j \geq 3$] have been neglected. The associated single degree of freedom potential energy functional is given by†

$$F(a) = A_2'(\lambda-\lambda_1)a^2 + A_2''(\lambda-\lambda_1)^2a^2 + A_3a^3 + A_3'(\lambda-\lambda_1)a^3 + A_4a^4 + Q_1(u_1)a$$
$$+ [Q_1'(u_1) - P_{11}(\varphi_0, \varphi_1')](\lambda-\lambda_1)a - P_{11}(\varphi_0, \varphi_2)a^2. \tag{23}$$

The quantities in equations (22) and (23) that require definition are as follows:

$$A_2' = P_2'[u_1], \tag{24}$$

$$A_2'' = P_2''[u_1] - P_2[\varphi_1'], \tag{25}$$

$$A_3 = P_3[u_1], \tag{26}$$

$$A_3' = P_3'[u_1] - P_{11}[\varphi_1', \varphi_2], \tag{27}$$

$$A_4 = P_4[u_1] - P_2[\varphi_2]. \tag{28}$$

It is to be understood that all functionals are evaluated at $\lambda = \lambda_1$. The unknown fields φ_0, φ_1' and φ_2 are defined by the linear relations given below:

$$P_{11}[\varphi_0, \xi] + Q_1[\xi] - \frac{Q_1[u_1]}{2T_2[u_1]} T_{11}[u_1, \xi] = 0, \tag{29a}$$

$$T_{11}[u_1, \varphi_0] = 0, \tag{29b}$$

$$P_{11}[\varphi_1', \xi] + P_{11}'[u_1, \xi] - \frac{P_2'[u_1]}{T_2[u_1]} T_{11}[u_1, \xi] = 0, \tag{30a}$$

$$T_{11}[u_1, \varphi_1'] = 0, \tag{30b}$$

$$P_{11}[\varphi_2, \xi] + P_{21}[u_1, \xi] - \frac{3P_3[u_1]}{2T_2[u_1]} T_{11}[u_1, \xi] = 0, \tag{31a}$$

$$T_{11}[u_1, \varphi_2] = 0. \tag{31b}$$

The functional T_2 is an arbitrary, positive-definite quadratic functional. It is used for defining orthogonality of displacement fields, but its choice does not affect the assessment of stability at the critical point.

The potential energy defined by equation (23) is an algebraic function of a single variable. Thus, the analysis is carried to completion by solution of the Euler equation

$$\frac{dF(a)}{da} = 0 \tag{32}$$

for "a" as a function of the applied load parameter. This result permits recovery of the actual displacement field as an explicit function of the applied load, as will be illustrated subsequently.

† It should be noted that in most applications of Koiter's work a lower order approximation consisting of the 1, 3, 5 and 6 terms in equation (23) is used.

3. THE ANALYSIS METHOD IN FINITE ELEMENT FORM

(a) *Notation*

The matrix representation and the attendant notation of a nonlinear finite element model are drawn from the work of Mallett and Marçal [13]. Specifically, the finite element representation of a geometrically nonlinear structure is taken to be given by,

$$\Pi(Q) = \lfloor Q \rfloor [\tfrac{1}{2}[K] + \tfrac{1}{6}[N1(Q)] + \tfrac{1}{12}[N2(Q, Q)]]\{Q\} - \lfloor Q \rfloor \{P\} \tag{33}$$

where

$\Pi(Q)$	is the total potential energy,
$\{Q\}, \lfloor Q \rfloor$	is the set of gridpoint displacement degrees of freedom,†
$\{P\}$	is the applied load vector associated with the displacement Q,
$[K]$	is the stiffness matrix,
$[N1(Q)]$	is a matrix whose terms depend linearly upon the displacement Q; it is referred to as the "first order incremental stiffness matrix",
$[N2(Q, Q)]$	is a matrix whose terms depend quadratically upon the displacement Q; it is referred to as the "second order incremental stiffness matrix".

The basis for the choice of this form of finite element representation is the inherent breakdown into the homogeneous energy contributions identified in equation (2), i.e.

$$\Pi_1(Q) = -\lfloor Q \rfloor \{P\}, \tag{34}$$

$$\Pi_2(Q) = \tfrac{1}{2}\lfloor Q \rfloor [K]\{Q\}, \tag{35}$$

$$\Pi_3(Q) = \tfrac{1}{6}\lfloor Q \rfloor [N1(Q)]\{Q\}, \tag{36}$$

$$\Pi_4(Q) = \tfrac{1}{12}\lfloor Q \rfloor [N2(Q, Q)]\{Q\}. \tag{37}$$

(b) *The perfect structure*

The Euler equations corresponding to equation (7) derive from

$$0 = -\lfloor \delta Q \rfloor \{P\} + \lfloor \delta Q \rfloor [K]\{Q\} + \tfrac{1}{2}\lfloor \delta Q \rfloor [N1(Q)]\{Q\} + \tfrac{1}{3}\lfloor \delta Q \rfloor [N2(Q, Q)]\{Q\}, \tag{38}$$

and linearization yields

$$\{0\} = -\{P\} + [K]\{Q\} \tag{39}$$

from which the linearized displacement $\{Q(\lambda)\}$ [equation (9)] corresponding to a load system $\lambda\{P_0\}$ is obtained as

$$\{Q(\lambda)\} = \lambda\{Q_0\} = [K]^{-1}\lambda\{P_0\}. \tag{40}$$

It follows immediately upon reference to equations (11) and (33) that the modification‡ $Q_1(Q)$ to obtain the potential energy function of the perfect structure [equation (10)] is

$$Q_1(Q) = \frac{\lambda^2}{2}\lfloor Q \rfloor [N1(Q_0)]\{Q_0\} + \frac{\lambda^3}{3}\lfloor Q \rfloor [N2(Q_0, Q_0)]\{Q_0\}. \tag{41}$$

† The brackets { } which identify Q as a column vector, and $\lfloor\ \rfloor$ which identify Q as a row vector are omitted where no ambiguity results.

‡ Note that Q_1 is scalar; all other Q's are displacement vectors.

(c) *The bifurcation point*

The finite element representation of the transition energy for the perfect structure [equation (13)], as defined by equations (14)–(17) is given by

$$P_1(q) \equiv 0, \tag{42}$$

$$P_2(q) = \tfrac{1}{2}\lfloor q \rfloor\big[[K] + \lambda[N1(Q_0)] + \lambda^2[N2(Q_0, Q_0)]\big]\{q\}, \tag{43}$$

$$P_3(q) = \tfrac{1}{6}\lfloor q \rfloor\big[[N1(q)] + 2[N2(q, \lambda Q_0)]\big]\{q\}, \tag{44}$$

$$P_4(q) = \tfrac{1}{12}\lfloor q \rfloor[N2(q, q)]\{q\}. \tag{45}$$

The particularization of equations (18) and (19) appropriate for determination of the critical buckling load level for the perfect structure is drawn from equation (43) as

$$P_{11}(\delta q, q) = 0 = \lfloor \delta q \rfloor\big[[K] + \lambda[(N1(Q_0)] + \lambda^2[N2(Q_0, Q_0)]\big]\{q\}. \tag{46}$$

The quadratic eigenvalue problem that results from equation (46) for arbitrary $\{\delta q\}$ is assumed to yield a smallest positive eigenvalue λ_1, which is distinct, and a unique eigenvector $\{q\} = \{q_1\}$.

(d) *Approximate behavior analysis*

The solution for the approximate behavior is essentially complete upon specification of the scalar coefficients in the single degree of freedom energy functional of equation (23). Thus, a direct translation of the functional notation for these coefficients into that of the finite element representation completes the analysis. As we are dealing with ordinary vectors we will use a functional $T_2(U)$ which corresponds to the usual scalar product, i.e.

$$T_2(q) = \tfrac{1}{2}\lfloor q \rfloor\{q\}. \tag{47}$$

Equations (30a, b) for $\{\varphi_1'\}$ now become

$$[K_T]\{\varphi_1'\} = \{r_1\}; \qquad \lfloor q_1 \rfloor\{\varphi_1'\} = 0 \tag{48}$$

where

$$[K_T] = [K] + \lambda_1[N1(Q_0)] + \lambda_1^2[N2(Q_0, Q_0)], \tag{49}$$

and

$$\{r_1\} = \frac{\lfloor q_1 \rfloor\big[[N1(Q_0)] + 2\lambda_1[N2(Q_0, Q_0)]\big]\{q_1\}}{\lfloor q_1 \rfloor\{q_1\}}\{q_1\} - \big[[N1(Q_0)] + 2\lambda_1[N2(Q_0, Q_0)]\big]\{q_1\}. \tag{50}$$

Equations (31a, b) for $\{\varphi_2\}$ now become

$$[K_T]\{\varphi_2\} = \{r_2\}; \qquad \lfloor q_1 \rfloor\{\varphi_2\} = 0 \tag{51}$$

where

$$\{r_2\} = \frac{\frac{1}{2}\lfloor q_1 \rfloor\big[[N1(q_1)] + 2\lambda_1[N2(q_1, Q_0)]\big]\{Q_0\}}{\lfloor q_1 \rfloor\{q_1\}}\{q_1\} - \tfrac{1}{2}\big[[N1(q_1)] + 2\lambda_1[N2(q_1, Q_0)]\big]\{q_1\}. \tag{52}$$

Equations (29a, b) for $\{\varphi_0\}$ now become

$$[K_T]\{\varphi_0\} = \{r_0\}; \quad \lfloor q_1 \rfloor \{\varphi_0\} = 0 \tag{53}$$

where

$$\{r_0\} = \frac{\frac{1}{2}\lambda_1^2 \lfloor q_1 \rfloor [[N1(Q_0)] + (2\lambda_1/3)[N2(Q_0, Q_0)]]\{Q_0\}}{\lfloor q_1 \rfloor \{q_1\}} \{q_1\}$$
$$-\tfrac{1}{2}\lambda_1^2 [[N1(Q_0)] + (2\lambda_1/3)[N2(Q_0, Q_0)]]\{Q_0\}. \tag{54}$$

The coefficients A_2', A_2'', etc., are given by equations (25)–(28), expressed in finite element notation:

$$A_2' = \frac{-1}{2\lambda_1} \lfloor q_1 \rfloor [[K] - \lambda_1^2 [N2(Q_0, Q_0)]]\{q_1\}, \tag{55}$$

$$A_2'' = \tfrac{1}{2}\lfloor q_1 \rfloor [N2(Q_0, Q_0)]\{q_1\} - \tfrac{1}{2}\lfloor \varphi_1' \rfloor [K_T]\{\varphi_1'\}, \tag{56}$$

$$A_3 = \tfrac{1}{6}\lfloor q_1 \rfloor [[N1(q_1)] + 2\lambda_1 [N2(Q_0, q_1)]]\{q_1\}, \tag{57}$$

$$A_3' = \tfrac{1}{3}\lfloor q_1 \rfloor [N2(Q_0, q_1)]\{q_1\} - \lfloor \varphi_1' \rfloor [K_T]\{\varphi_2\}, \tag{58}$$

$$A_4 = \tfrac{1}{12}\lfloor q_1 \rfloor [N2(q_1, q_1)]\{q_1\} - \tfrac{1}{2}\lfloor \varphi_2 \rfloor [K_T]\{\varphi_2\}. \tag{59}$$

4. APPLICATIONS

(a) *Structure with one degree of freedom*

The simple structure illustrated in Fig. 2(a) is considered in order to make transparently clear the modified structure method of analysis and its relation to the method of Koiter. The structure consists of two rigid, pin-jointed bars whose common end-point is supported laterally by a nonlinear spring. The load–displacement behavior in the neighborhood of the snap-through buckling load is sought. This problem is a simplified version of a problem considered previously by Hoff [15, 16]. It can evidently be interpreted as a perfect structure, Fig. 2(b), which has an initial small imperfection "e". Thus, the conventional method of

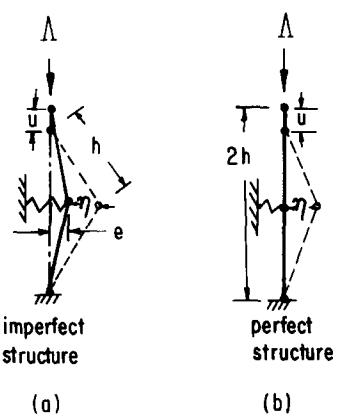


FIG. 2. Description of column model.

Koiter is applicable, and this method is presented first in what follows. Then, the modified structure method, developed in the present paper for application to problems without a recognizable adjacent perfect structure, is applied.

The fundamental equilibrium path for the perfect structure, Fig. 2(b), is trivial (zero). The force exerted by the lateral spring is taken to be of the form

$$F(\eta) = K_1\eta + K_2\eta^2 \tag{60}$$

where η is the deflection. Then, the potential energy function for the perfect structure is written as

$$\Pi(\eta) = \tfrac{1}{2}K_1\eta^2 + \tfrac{1}{3}K_2\eta^3 - \Lambda u. \tag{61}$$

It is convenient to introduce nondimensional quantities defined by

$$\zeta = \frac{\eta}{h}, \qquad \pi = \frac{\Pi}{K_1h^2}, \tag{62a, b}$$

$$k = \frac{K_2h}{K_1}, \qquad \lambda = \frac{2\Lambda}{K_1h}. \tag{62c, d}$$

The nondimensional total potential energy function takes the form

$$\pi(\zeta) = \frac{1}{2}\zeta^2 + \frac{1}{3}k\zeta^3 - \frac{\lambda}{2}\left(\frac{u}{h}\right). \tag{63}$$

From the Pythagoran theorem and a series expansion, we obtain

$$\frac{1}{2}\left(\frac{u}{h}\right) = \frac{1}{2}\zeta^2 - \frac{1}{8}\zeta^4 + \ldots. \tag{64}$$

Because the fundamental solution is zero, the total potential energy is also the energy of transition from the fundamental state, and the result of substituting equation (64) into (63) is expressible as

$$\pi(\zeta) = P(\zeta) = P_2(\zeta) + P_3(\zeta) + P_4(\zeta) + \ldots \tag{65}$$

where

$$P_2(\zeta) = \tfrac{1}{2}\zeta^2 - \tfrac{1}{2}\lambda\zeta^2, \tag{66}$$

$$P_3(\zeta) = \tfrac{1}{3}k\zeta^3, \tag{67}$$

$$P_4(\zeta) = \tfrac{1}{8}\lambda\zeta^4, \tag{68}$$

etc.

In the following, only the first two terms, $P_2(\zeta)$ and $P_3(\zeta)$, are retained in the interest of algebraic simplicity. Since the same energy functional is used for both the Koiter method and the modified structure method, this approximation will not affect the comparison.

(i) *Koiter's method.* The analysis is carried forward by using equation (18) in conjunction with equation (66) to determine the bifurcation load λ_1:

$$P_{1\,1}(\zeta, \delta\zeta) = 0 = (\delta\zeta)(1-\lambda)\zeta \tag{69}$$

yields $\lambda_1 = 1$, and the mode shape at bifurcation can be taken as $\zeta_1 = 1$.

Following the method of Koiter, a first approximation to the energy function in the near post-buckling regime is given by the substitution $\zeta = a\zeta_1$ into equation (65):

$$\pi(a\zeta_1) = P_2(a) + P_3(a); \tag{70}$$

or

$$\pi(a) = \tfrac{1}{2}a^2(\lambda - \lambda_1) + \tfrac{1}{3}ka^3. \tag{71}$$

The approximation to post-buckling behavior is obtained by requiring $d\pi(a)/da = 0$, which gives

$$\lambda = 1 - ka. \tag{72}$$

Consider now that the actual structure is the perfect structure specified to have a misalignment "e". In terms of a nondimensional imperfection $\varepsilon = e/h$, the appropriately modified form of equation (64) is

$$\frac{1}{2}\frac{u}{h} = \frac{1}{2}\zeta^2 - \frac{1}{8}\zeta^4 + \varepsilon\zeta + \ldots. \tag{73}$$

This result leads to a corresponding approximation of the modification to the energy function $\pi(\zeta)$, equation (65), of the form

$$Q_1(\zeta) = -\lambda\varepsilon\zeta. \tag{74}$$

Thus, for the imperfect structure, equation (71) with the modification (74) becomes

$$\pi(a) = -\tfrac{1}{2}(1-\lambda)a^2 + \tfrac{1}{3}ka^3 - \lambda\varepsilon a \tag{75}$$

and the approximate equilibrium path is determined by

$$\frac{d\pi(a)}{da} = -(1-\lambda)a + ka^2 - \lambda\varepsilon = 0. \tag{76}$$

This approximate solution for behavior in the vicinity of buckling load is illustrated graphically in Fig. 3 as a plot of load vs. displacement for the specified value of $k\varepsilon$. The displacement a^* at the maximum load λ^* is obtained by solving equation (76) explicitly for λ and differentiating to find a^* such that $d\lambda/da = 0$ for $a = a^*$:

$$-(1-\lambda) + 2ka^* = 0. \tag{77}$$

Substitution for a^* into equation (76) yields an equation for λ^*:

$$-\frac{(1-\lambda^*)^2}{4k} - \lambda^*\varepsilon = 0. \tag{78}$$

For $\lambda^* \approx 1$, equation (78) is alternatively expressible as

$$\lambda^* = 1 - 2(-k\varepsilon)^{\frac{1}{2}}, \qquad k\varepsilon \leq 0. \tag{79}$$

For $k\varepsilon > 0$, there is no maximum load. Equation (78), illustrated in Fig. 4 as a plot of λ^* vs. $-k\varepsilon$, describes the imperfection sensitivity of the structure.

(ii) *Modified structure method.* The total potential energy of the imperfect structure [equation (2)] is written using equations (63) and (73) as

$$\pi(\zeta) = \tfrac{1}{2}\zeta^2 + \tfrac{1}{3}k\zeta^3 - \lambda(\tfrac{1}{2}\lambda^2 + \varepsilon\zeta). \tag{80}$$

Invoking equation (8), which draws upon only the first and last term in equation (80), we obtain a result† in the form of equation (9):

$$\zeta_l = \lambda\varepsilon. \tag{81}$$

This solution is, on the one hand, a linear approximation to the behavior of the actual structure. On the other hand, in accord with equation (10), it is also the exact behavior of the modified structure defined by the modified potential $\bar{\pi}(\zeta)$:

$$\bar{\pi}(\zeta) = \pi(\zeta) - Q_1(\zeta) \tag{82}$$

where $\pi(\zeta)$ is given by equation (80), and the modifying term is

$$Q_1(\zeta) = -\lambda^2\varepsilon\zeta + k\lambda^2\varepsilon^2\zeta. \tag{83}$$

Before calculation of the bifurcation load of this perfect structure, we express the contributions to the transition energy of equation (13) for this example as†

$$P_2(\Delta\zeta) = \tfrac{1}{2}(1-\lambda)\Delta\zeta^2 + k\lambda\varepsilon\Delta\zeta^2, \tag{84}$$

$$P_3(\Delta\zeta) = \tfrac{1}{3}k\Delta\zeta^3. \tag{85}$$

Now, use of equation (18) in conjunction with equation (84) yields,

$$P_{11}(\delta\zeta, \Delta\zeta) = 0 = [(1-\lambda) + 2k(\lambda\varepsilon)](\Delta\zeta)(\delta\zeta). \tag{86}$$

It follows that the bifurcation load of the perfect structure is

$$\lambda_1 = (1-2k\varepsilon)^{-1} = 1 + O(\varepsilon). \tag{87}$$

The associated buckling mode is denoted by $\Delta\zeta_1 = 1$.

The post-buckling behavior is sought through equation (23) which, to a first approximation, is expressed in the form

$$F(a) = A_2'(\lambda-\lambda_1)a^2 + A_3a^3 + A_1a. \tag{88}$$

For this example,

$$A_2' = \left.\frac{\partial P_2}{\partial\lambda}\right|_{\lambda=\lambda_1} = -(\tfrac{1}{2}-k\varepsilon), \tag{89a}$$

$$A_3 = P_3|_{\lambda=\lambda_1} = \tfrac{1}{3}k, \tag{89b}$$

$$A_1 = Q_1|_{\lambda=\lambda_1} = -\lambda_1^2\varepsilon(1-k\varepsilon). \tag{89c}$$

The equation of equilibrium is then

$$\frac{\partial F(a)}{\partial a} = 2A_2'(\lambda-\lambda_1)a + 3A_3a^2 + A_1 = 0 \tag{90}$$

or, upon substitution from equations (89a, b, c),

$$0 = -2(\tfrac{1}{2}-k\varepsilon)(\lambda-\lambda_1)a + ka^2 - \lambda_1^2\varepsilon(1-k\varepsilon). \tag{91}$$

† To correlate with the notation of Section 2, regard ζ_l as U_l, and $\Delta\zeta$ as u.

The graph of this approximate solution for the load–displacement behavior is shown in Fig. 3. The maximum load $\bar{\lambda}^*$ is derived from the condition $d\lambda/da = 0$ and is

$$\bar{\lambda}^* = \lambda_1 - \frac{(3A_3A_1)^{\frac{1}{2}}}{A_2'}. \tag{92}$$

Substituting from equations (89a, b, c), we have then

$$\bar{\lambda}^* = \lambda_1 - 2\lambda_1(-k\varepsilon)^{\frac{1}{2}}\frac{(1-k\varepsilon)^{\frac{1}{2}}}{1-2k\varepsilon} \tag{93}$$

or

$$\bar{\lambda}^* = 1 - 2(-k\varepsilon)^{\frac{1}{2}} + O(\varepsilon). \tag{94}$$

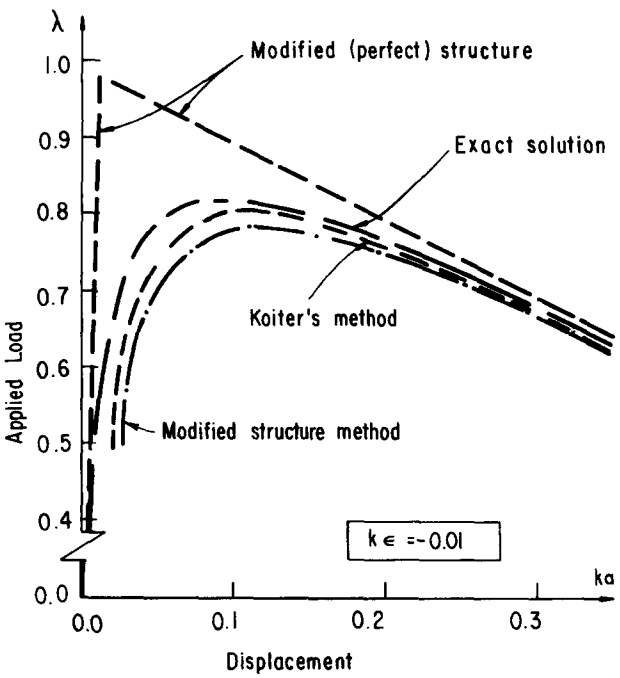


FIG. 3. Behavior of column model.

It should be noted that λ_1 and $\bar{\lambda}^*$ do not depend on k and ε independently but only on the product $k\varepsilon$.

The values of λ^*, $\bar{\lambda}^*$ and λ_1 are plotted in Fig. 4 against $-k\varepsilon$. The agreement between λ^* in equation (79), and $\bar{\lambda}^*$ in equation (93), is very good even when the linear eigenvalue analysis gives a bifurcation load (λ_1) for the modified structure which is 50 per cent higher than the buckling load of the actual structure. Equations (79) and (94) show that the Koiter method and the modified structure method are equivalent asymptotically for ε tending toward zero, and Fig. 4 shows that agreement between the results of the two methods is good over a wide range. Figure 3 shows that both Koiter's method and the modified

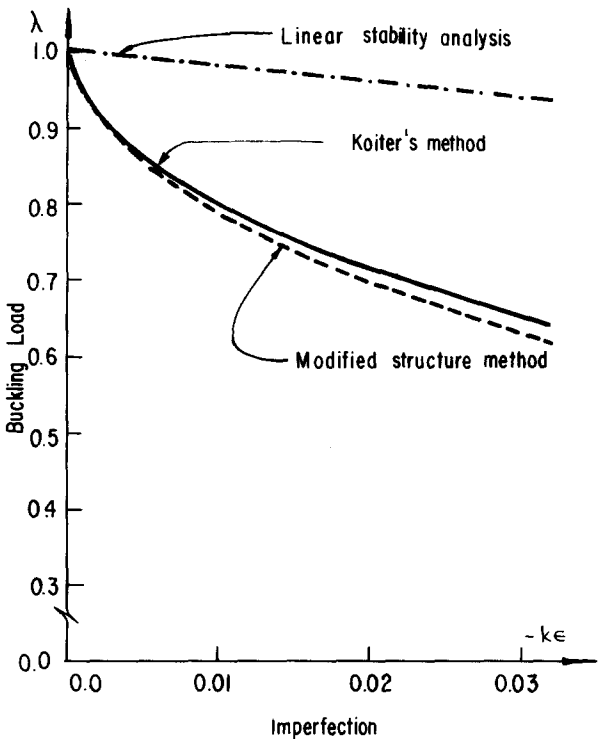
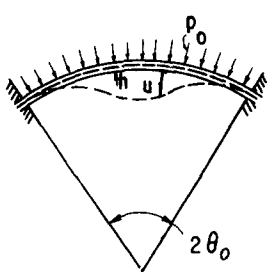


FIG. 4. Imperfection sensitivity of the column model.

structure method are useful not only for calculation of critical loads but also for prediction of load displacement behavior.

(b) *Shallow circular arch*

The problem of the shallow, clamped, circular arch under uniform lateral pressure (see Fig. 5) is used by Kerr and Soifer [2] to demonstrate the short-comings of a linear



$R = 10$ in $\quad h = 0.0684$ in

$\theta_0 = 15°$

$EA = 2.056 \times 10^6$ lb, $EI = 796$ lb in^2

FIG. 5. Circular arch under uniform pressure.

eigenvalue stability analysis. The arch exhibits nonlinear, symmetric pre-buckling behavior. The minimum buckling load is of the bifurcation type associated with asymmetric displacements. If the arch is forced to remain symmetric, it will snap-through at a somewhat higher load than the bifurcation buckling load.

The arch chosen in [2] for calculation is quite shallow, and a linear analysis in [2] yields a 10 per cent error for the bifurcation load and 50 per cent error for the snap-through load. The modified structure method combined with a finite element idealization of the arch was applied to each case, and it yielded more information with better accuracy.

For the case of the bifurcation load, we start with a perfect structure but a nonlinear one. The present method creates an adjacent perfect linear structure. When the approximate behavior is analyzed with the lowest order (and the generally used) Koiter's approximation, no correction at all is possible. However, the application of the more refined analysis represented by equation (23) yields a correction. Instead of 10 per cent error in the linear analysis, the error reduces to 3 per cent.

The snap-through case is interesting as an example for the modified structure method because there is no obvious adjacent perfect structure. The 50 per cent error of the linearized eigenvalue stability analysis is reduced to 5 per cent by the present method.

The results pertaining to the different buckling loads are summarized in Table 1. Figure 6 shows the approximate behavior of the arch derived from the present analysis as

TABLE 1. CIRCULAR ARCH BUCKLING LOADS $\bar{p}$

Case	Buckling load†		
	Exact [2]	Linear stability analysis	Modified structure method
Bifurcation	1·91	2·14	1·86
Snap-through	2·27	3·45	2·17

† Nondimensional pressure $\bar{p} = p_0 \frac{R^2 h}{EI}$.

compared to the exact solution obtained in [2] and also to the direct nonlinear solution that we obtained by the finite element idealization. We see that, even in this case, where a linear eigenvalue prediction is 50 per cent higher than the actual buckling load, the modified structure method yields good results. It is also important to note that the modified structure method yields accurate load-displacement behavior in the vicinity of the buckling load for this example.

5. CONCLUSIONS

The principal conceptual contribution of the modified structure method of analysis is the adaption of the theory of Koiter [3] to treat moderately nonlinear structures that exhibit snap-through buckling behavior. Unlike methods previously based upon [3], the modified structure method of analysis produces a linearized fundamental equilibrium path. This characteristic is an important factor to the utility of the method.

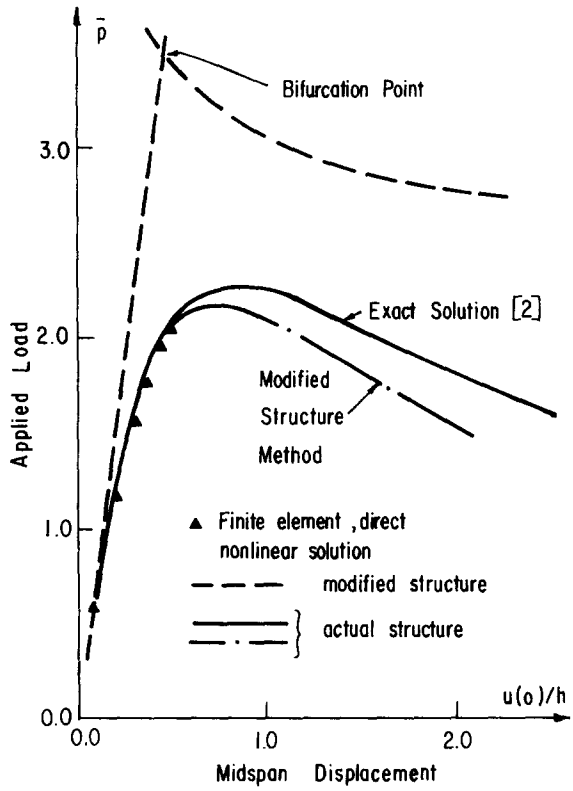


FIG. 6. Behavior of the circular arch.

Particularly significant is that, although the modified structure method of analysis is developed in a functional form, a highly systematic procedure is obtained by the subsequent introduction of matrix notation in conjunction with finite element idealization. The broad applicability of the finite element approach with regard to the type of structures considered to take the best advantage of the broad applicability of the modified structure method of analysis with regard to types of nonlinear behavior.

The principal finite element matrices employed in the modified structure method of analysis are presently in use in automated systems for linear (eigenvalue) stability and nonlinear behavior analysis. Thus, the modified structure analysis can be implemented readily. The computational effort of a modified structure analysis is comparable to that of a linear stability analysis, while the information obtained is comparable to that of a direct nonlinear solution for behavior.

The asymptotic character of the theory of [3] is preserved by the modified structure method of analysis. The former is exact for vanishingly small imperfection, while the latter is similarly exact for vanishing small nonlinearity of the fundamental equilibrium path. Imperfections of the usual type (manufacturing variances, etc.) are not considered explicitly in the present modified structure formulation under the assumption that the finite element idealization includes account of such variations.

The numerical results for the circular arch illustrate the applicability of the modified structure method to the analysis of snap-through buckling behavior. The value of the second-order approximation provided by the modified structure method is clarified also. A more comprehensive numerical evaluation of the modified structure method of analysis is planned for future publication [14] (see also [17]).

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Абстракт—Применяется способ Койтера к анализу поведения нагрузкаперемещение, вблизи точки разветвления устойчивости, для анализа поведения вблизи такой самой точки разветвления для конструкции, у которой происходит прощелкивание. Основным пончтцем является обсуждение нелинейностей до момента потери устойчивости, в смысле начальных неправильностей, для определенной, идеальной конструкции, наеываемой модифицированной. Для этого случая учитывается асимптотический характер способо Койтера. Разработка дается в функциональной записи, и затем, она сводится к матричному процессу, основанному на идеализации конечного элемента. Даются численные результаты для поведения нагрузка-перемещение для круговой арки.